**Prompt:** Prove

$$\lim_{x\to\infty} \frac{x}{7+3x} = \frac{1}{3}.$$

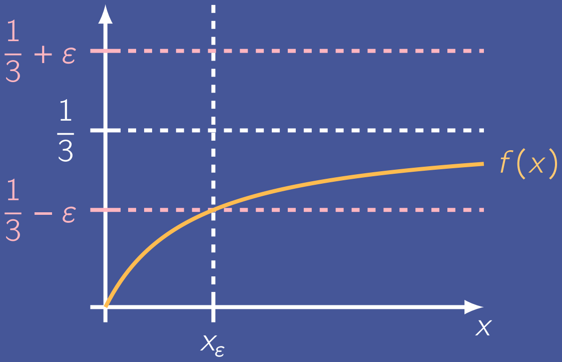
**Step 1:** Identify relevant definitions/theorems.

- **Definition of Function Limit at $\infty$**

  We say $f(x) \to L$ as $x \to \infty$ provided, for every $\varepsilon > 0$, there is $x_\varepsilon > 0$ such that

  $$x \geq x_\varepsilon \implies |f(x) - L| \leq \varepsilon.$$

**Step 2:** Illustrate the problem.




$$\left| f(x) - \tfrac{1}{3} \right| \leq \varepsilon, \quad \text{for all } x \geq x_\varepsilon$$

**Step 3:** Do scratch work for key equations.

To establish convergence, we need

$$\left| f(x) - \frac{1}{3} \right| \leq \varepsilon, \quad \text{for large } x.$$

Note

$$f(x) = \frac{x}{7+3x} = \frac{1}{7/x+3} \leq \frac{1}{3}, \quad \text{for all } x > 0.$$

Additionally,

$$\begin{aligned} f(x) > \frac{1}{3} - \varepsilon \quad &\iff \quad \frac{1}{7/x+3} > \frac{1-3\varepsilon}{3} \\ &\iff \quad 3 > \left(\frac{7}{x} + 3\right)(1-3\varepsilon) \\ &\iff \quad 9\varepsilon > \frac{7-21\varepsilon}{x} \\ &\iff \quad x > \frac{7-21\varepsilon}{9\varepsilon} = \frac{7}{9}\left(\frac{1}{\varepsilon} - 3\right). \end{aligned}$$

**Step 4:** Draft a proof.

Define the function $f$ by $f(x) = x/(7+3x)$ for $x > 0$. Let $\varepsilon > 0$ be given. To prove $\lim_{x\to\infty} f(x) = 1/3$, it suffices to verify there is $x_\varepsilon > 0$ such that

$$x \geq x_\varepsilon \quad \Longrightarrow \quad \left| f(x) - \frac{1}{3} \right| \leq \varepsilon.$$

Set $\delta = \min\{\varepsilon, 1/5\}$ and

$$z = \frac{7}{9}\left(\frac{1}{\delta} - 3\right),$$

noting $z > 0$ since $1/\delta - 3 \geq 5 - 3 = 2$. Thus,

$$x \geq z \quad \Longrightarrow \quad f(x) = \frac{x}{7+3x} = \frac{1}{7/x+3} \leq \frac{1}{3}.$$

Since

$$\frac{7}{z} = \frac{7}{\frac{7}{9}\left(\frac{1}{\delta} - 3\right)} = \frac{9}{\frac{1}{\delta} - 3} = \frac{9\delta}{1 - 3\delta},$$

it follows for $x \geq z$ that

$$f(x) = \frac{1}{7/x + 3} \geq \frac{1}{\frac{9\delta}{1-3\delta} + 3} = \frac{1 - 3\delta}{3} = \frac{1}{3} - \delta.$$

Hence, by the above results,

$$x \geq z \quad \Longrightarrow \quad \frac{1}{3} - \varepsilon \leq \frac{1}{3} - \delta \leq f(x) \leq \frac{1}{3} \leq \frac{1}{3} + \varepsilon,$$

and so the desired statement holds with $x_\varepsilon = z$. ■

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