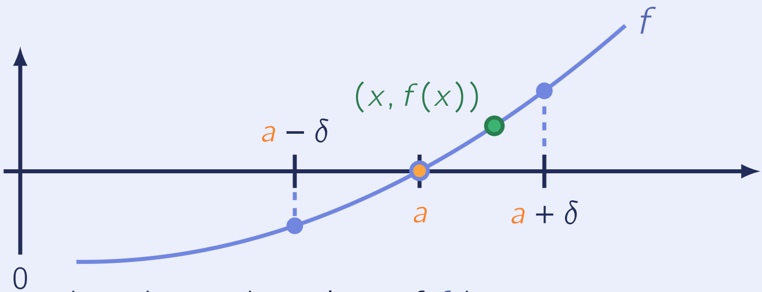
Prove if f is differentiable with $f(a) = 0$ and $f'(a) \neq 0$,

then f is nonzero in an interval about a (except at a).



key observation: slope of f is nonzero near a

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Proof

We first show f' is nonzero near a (Step 1) and then apply the mean value theorem with this fact (Step 2).

Step 1. Since $f'(a) \neq 0$, there is $\delta > 0$ such that

$$|x - a| \leq \delta \quad \Longrightarrow \quad |f'(x) - f'(a)| \leq \frac{|f'(a)|}{2}.$$

By the reverse triangle inequality,

$$|f'(a)| - |f'(x)| \leq |f'(x) - f'(a)|, \quad \text{for all } x.$$

Together, these results imply

$$|x - a| \leq \delta \quad \Longrightarrow \quad |f'(x)| \geq |f'(a)| - \frac{|f'(a)|}{2} > 0.$$

Thus, $f'(x)$ is nonzero for all $x \in [a - \delta, a + \delta]$.

Proof Continued

Step 2. Let $x \in [a-\delta, a) \cup (a, a+\delta]$ be given, and note $x \neq a$. By the mean value theorem, there is ξ_x between x and a such that

$$f'(\xi_x)(x-a) = f(x) - f(a) = f(x),$$

where the final equality holds since $f(a) = 0$. Since ξ_x is between x and a, it is in the interval $[a-\delta, a+\delta]$, and so $f'(\xi_x) \neq 0$ by Step 1. Thus,

$$f(x) = \underbrace{f'(\xi_x)}_{\neq 0} \cdot \underbrace{(x-a)}_{\neq 0} \neq 0.$$

Because x was arbitrarily chosen in the interval $[a-\delta, a) \cup (a, a+\delta]$, the result follows. ∎

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