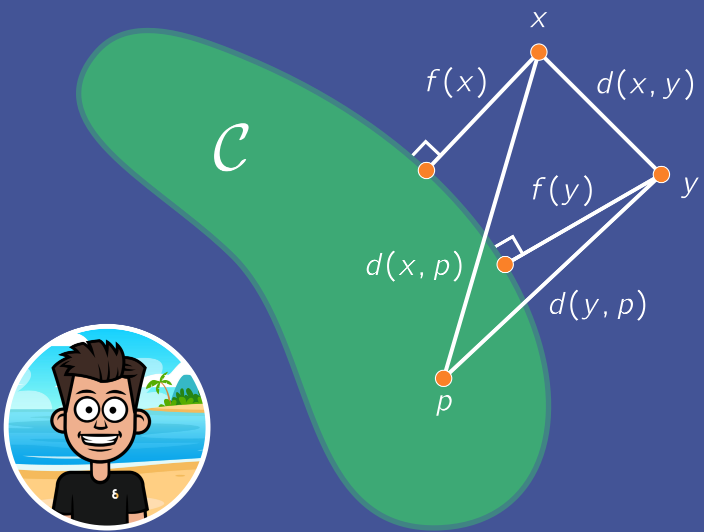
Distance to Set

is

Uniformly Continuous



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Prompt

Define the distance function $f\colon \mathcal{X} \to \mathbb{R}$ for a set $\mathcal{C}$ by

$$f(x) = \inf_{p \in \mathcal{C}} d(x, p).$$

Prove f is uniformly continuous.

Intuition for Scratch Work

- Draw a diagram (like on previous slide).
- Recall definition of uniform continuity:

  Given $\varepsilon > 0$, there is $\delta > 0$ such that, for all x and y,

  $$d(x, y) \leq \delta \quad \Longrightarrow \quad |f(x) - f(y)| \leq \varepsilon.$$

- Note any relations between $f(x)$ and $f(y)$:
  - $f(x) \leq d(x, p)$ and $f(y) \leq d(y, p)$,
  - $d(x, p) \leq d(x, y) + d(y, p)$ by triangle inequality.

Proof

Let $\varepsilon > 0$ and $x, y \in \mathcal{X}$ be given. To prove f is uniformly continuous, it suffices to show

$$d(x,y) \leq \varepsilon \quad \Longrightarrow \quad |f(x) - f(y)| \leq \varepsilon.$$

By the definition of f and the triangle inequality,

$$f(x) \leq d(x,p) \leq d(x,y) + d(y,p), \quad \text{for all } p \in \mathcal{C}.$$

This implies

$$f(x) - d(x,y) \leq d(y,p), \quad \text{for all } y \in \mathcal{C},$$

i.e. $f(x) - d(x,y)$ is a lower bound on the distance between y and each point p in $\mathcal{C}$.

Proof

Since the infimum is the greatest lower bound,

$$f(x)-d(x,y) \le f(y) \quad\Longrightarrow\quad f(x)-f(y) \le d(x,y).$$

By analogous argument,

$$f(y)-f(x) \le d(y,x) = d(x,y).$$

Together, the last two inequalities imply

$$d(x,y) \le \varepsilon \quad\Longrightarrow\quad |f(x)-f(y)| \le d(x,y) \le \varepsilon,$$

as desired. ∎

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