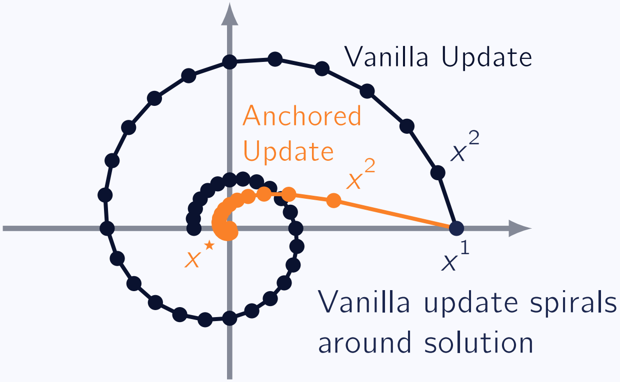
Adding an Anchor ⚓

can Speed Up Your Algorithm



Vanilla update spirals around solution

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Fixed Point Iteration

Many algorithms can be written using an operator, denoted here by T. Given input x^1, each update is

$$x^{k+1} = T(x^k).$$

For "nice" operators T, the sequence $\{x^k\}$ converges to a limit $x^\star$ for which $x^\star = T(x^\star)$.

Example: Under standard assumptions, the problem

$$\min_{x \in \mathcal{C}} f(x)$$

can be solved with the projected gradient operator

$$T(x) = \mathrm{proj}_{\mathcal{C}}(x - \alpha \nabla f(x)),$$

with $\alpha > 0$ a step size and $\mathrm{proj}_{\mathcal{C}}$ the projection onto $\mathcal{C}$.

Anchoring via Halpern

Given input x^1, each update for Halpern iteration is[†]

$$x^{k+1} = \frac{1}{n} \cdot x^1 + \left(1 - \frac{1}{n}\right) \cdot T(x^k).$$

Each update x^{k+1} is an average of x^1 and $T(x^k)$.

Theorem: When T has a fixed point and is "nice,"[‡] the sequence $\{x^k\}$ generated by Halpern converges to a limit $x^\star$ for which $x^\star = T(x^\star)$. Moreover,

$$\|x^{k+1} - x^k\| = \mathcal{O}(1/k^2).$$

Note: Without anchoring, we may only be able to guarantee $\mathcal{O}(1/k)$ rate of convergence.

[†] A special case of a more general formula is shown here.

[‡] That is, $\|T(x) - T(y)\| \leq \|x - y\|$ for all x and y

Example

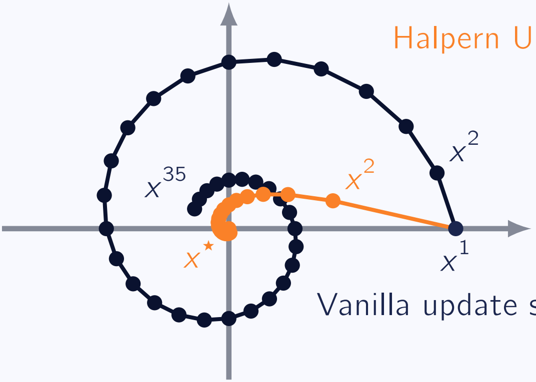
Suppose the update operator T is a rotation in the 2D plane by θ that is scaled by $0 < \lambda < 1$, *i.e.*

$$T(x) = \lambda \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Here $x^\star$ is the origin.

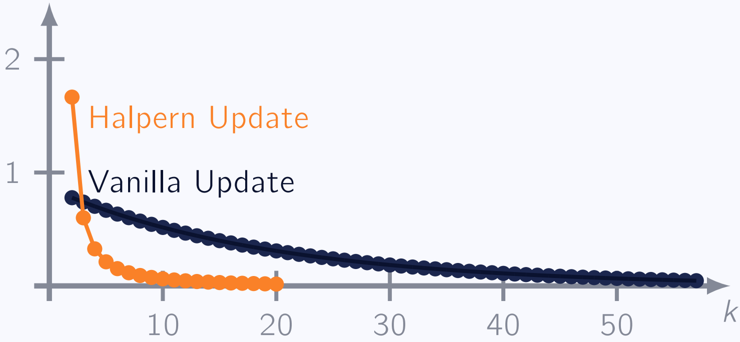
Vanilla Update: $x^{k+1} = T(x^k)$

Halpern Update

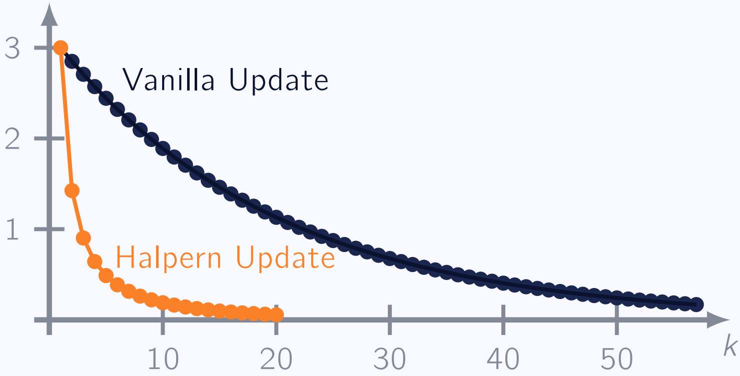


Vanilla update spirals around $x^\star$

Example Continued



Update Residual $\|x^{k+1} - x^k\|$ vs. iteration k



Estimate Error $\|x^k - x^\star\|$ vs. iteration k

Takeaways

- Anchoring is trivial to implement

  (weighted average with x^1 and $T(x^k)$)

- Anchoring can improve convergence rate

  (*i.e.* go from $\mathcal{O}(1/k)$ to $\mathcal{O}(1/k^2)$)

- Anchoring can mitigate oscillatory behavior

- Anchoring is distinct from Nesterov acceleration

Reference: Example drawn from *Exact Optimal Accelerated Complexity for Fixed-Point Iterations*

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